CHAPTER 3 REVIEW

Throughout this review note,

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

denotes an $n \times n$ square matrix.

Part I. Cofactor expansion

1. When using cofactor expansion, we always choose to expand along the row or column containing the most zeros.

2. Sometimes it is much easier to use elementary row operations to erase all but one element in one of the rows before using cofactor expansion.

3. $\det A = \det A^T$.

4. Because of **3**, we may also use elementary column operations (multiplying one column by a constant, swaping two columns, substracting one column by a constant multiple of another column) to erase all but one element in one of the columns before using cofactor expansion.

Example 0.1. *Find the determinant of*

$$A = \begin{bmatrix} 3 & 5 & 6 \\ 2 & 4 & 3 \\ 2 & 3 & 5 \end{bmatrix}$$

Solution.

$$\det \begin{bmatrix} 3 & 5 & 6 \\ 2 & 4 & 3 \\ 2 & 3 & 5 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 3 \\ 2 & 4 & 3 \\ 2 & 3 & 5 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & -3 \\ 0 & 1 & -1 \end{bmatrix}$$
$$= \det \begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix} = 1$$

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Example 0.2. *Find the determinant of*

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ k & 2 & 4 & 10 \\ k & 1 & 1 & 1 \\ -k & 1 & k & k^2 \end{bmatrix}$$

Solution.

$$\begin{aligned} \det A &= \det \begin{bmatrix} 2 & 4 & 10 \\ 1 & 1 & 1 \\ 1 & k & k^2 \end{bmatrix} \\ &= \det \begin{bmatrix} 2 & 2 & 8 \\ 1 & 0 & 0 \\ 1 & k-1 & k^2-1 \end{bmatrix} \\ &= \det \begin{bmatrix} 2 & 8 \\ k-1 & k^2-1 \end{bmatrix} = 2k^2 - 8k + 6. \end{aligned}$$

In the third line above, we have added -1 times the first column to the second and third column. ◀

Part II. Adjoint matrix

1. Adjoint matrix is particularly useful in computing the inverse of 2×2 matrix, that is,
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$
2. Usually, it is not suggested to use adjoint to compute the inverse of a large matrix, say more than 3×3. But you are asked to calculate only several terms in A^{-1}.

Example 0.3. *Find the sum of the entries in the second row of* A^{-1}*, where*

$$A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix}.$$

Solution. It is easy to get $\det A = -1$. We need to compute the entries in the second row in the adjoint of A, that is

$$C_{12} = -\det \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} = 1, \quad C_{22} = -1, \quad C_{32} = -1.$$

So the sum of the entries in the second row of A^{-1} is 1. ◀

Part III. Cramer's rule

1. Cramer's rule is particularly useful in solving linear system containing functions

Example 0.4. *See Example 2 in*

http://www.math.purdue.edu/~shao92/documents/Examples_7.6.pdf

Part IV. Relationship to other materials. You can find a list of equivalent properties to $\det A \neq 0$ in

http://www.math.purdue.edu/~shao92/documents/Linear%20Algebra%20Review.pdf